DP IB Maths: AA HL

Your notes

5.4 Further Integration

Contents

Your notes

5.4.1 Integrating Special Functions

Integrating Trig Functions

How do I integrate sin, cos and sec^2?

- The **antiderivatives** for **sine** and **cosine** are

$$\int \sin x \, dx = -\cos x + c$$

$$\int \cos x \, dx = \sin x + c$$

where c is the **constant** of **integration**

- Also, from the **derivative** of $\tan x$

$$\int \sec^2 x \, dx = \tan x + c$$

- The **derivatives** of $\sin x$, $\cos x$ and $\tan x$ are in the **formula booklet**
  - so these **antiderivatives** can be easily deduced
- For the **linear** function $ax + b$, where a and b are constants,

$$\int \sin(ax + b) \, dx = -\frac{1}{a}\cos(ax + b) + c$$

$$\int \cos(ax + b) \, dx = \frac{1}{a}\sin(ax + b) + c$$

$$\int \sec^2(ax + b) \, dx = \frac{1}{a}\tan(ax + b) + c$$

- For **calculus** with **trigonometric** functions **angles must** be **measured** in **radians**
  - Ensure you know how to change the angle mode on your GDC

Examiner Tip

- The formula booklet can be used to find antiderivatives from the derivatives
  - Make sure you have the page with the section of standard derivatives open
  - Use these backwards to find any antiderivatives you need
  - Remember to add 'c', the constant of integration, for any indefinite integrals

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Worked example

a) Find, in the form $F(x) + c$, an expression for each integral

i. $\int \cos x \, dx$

ii. $\int \sec^2\left(3x - \frac{\pi}{3}\right) dx$

i. $\int \cos x \, dx = \sin x + c$

ii. $\int \sec^2\left(3x - \frac{\pi}{3}\right) dx = \frac{1}{3}\tan\left(3x - \frac{\pi}{3}\right) + c$

(Linear function $ax+b$)

b) A curve has equation $y = \int 2\sin\left(2x + \frac{\pi}{6}\right) dx$. The curve passes through the point with coordinates $\left(\frac{\pi}{3}, \sqrt{3}\right)$.

Find an expression for y.

Your notes

$$y = 2\int \sin\left(2x + \frac{\pi}{6}\right) dx$$

$$y = 2\left[-\frac{1}{2}\cos\left(2x + \frac{\pi}{6}\right)\right] + c$$

At $x = \frac{\pi}{3}, y = \sqrt{3},$ $\quad \sqrt{3} = -\cos\left(\frac{2\pi}{3} + \frac{\pi}{6}\right) + c$

$$c = \cos\left(\frac{5\pi}{6}\right) + \sqrt{3}$$

$$c = \frac{\sqrt{3}}{2}$$

$$\therefore y = \frac{\sqrt{3}}{2} - \cos\left(2x + \frac{\pi}{6}\right)$$

Integrating e^x & 1/x

How do I integrate exponentials and 1/x?

- The **antiderivatives** involving e^x and $\ln x$ are

$$\int \mathrm{e}^x \, \mathrm{d}x = \mathrm{e}^x + c$$

$$\int \frac{1}{x} \, \mathrm{d}x = \ln|x| + c$$

where c is the **constant** of **integration**

  - These are given in the **formula booklet**
- For the **linear** function $(ax + b)$, where a and b are constants,

$$\int \mathrm{e}^{ax+b} \, \mathrm{d}x = \frac{1}{a}\mathrm{e}^{ax+b} + c$$

$$\int \frac{1}{ax+b} \, \mathrm{d}x = \frac{1}{a}\ln|ax+b| + c$$

- It follows from the last result that

$$\int \frac{a}{ax+b} \, \mathrm{d}x = \ln|ax+b| + c$$

  - which can be deduced using **Reverse Chain Rule**
- With **ln,** it can be useful to write the constant of integration, c, as a logarithm
  - using the laws of logarithms, the answer can be written as a single term
  - $\int \frac{1}{x} \, \mathrm{d}x = \ln|x| + \ln k = \ln k|x|$ where k is a constant
  - This is similar to the special case of **differentiating** $\ln(ax+b)$ when $b = 0$

Examiner Tip

- Make sure you have a copy of the formula booklet during revision but don't try to remember everything in the formula booklet
  - However, do be familiar with the **layout** of the formula booklet
    - You'll be able to quickly locate whatever you are after
    - You do not want to be searching every line of every page!
  - For formulae you think you have remembered, use the booklet to double-check

Your notes

Your notes

Worked example

A curve has the gradient function $f'(x) = \dfrac{3}{3x+2} + e^{4-x}$.

Given the exact value of $f(1)$ is $\ln 10 - e^3$ find an expression for $f(x)$.

$$f(x) = \int \left(\frac{3}{3x+2} + e^{4-x} \right) dx$$

$$f(x) = 3\int \frac{1}{3x+2}\, dx + \int e^{4-x}\, dx$$

$$= 3\left[\frac{1}{3}\ln|3x+2|\right] - e^{4-x} + c$$

$$f(1) = \ln 10 - e^3, \quad \ln|3\times 1+2| - e^{4-1} + c = \ln 10 - e^3$$

$$\therefore c = \ln 10 - \ln 5$$

$$c = \ln\left(\frac{10}{5}\right) = \ln 2$$

$$\therefore f(x) = \ln|3x+2| - e^{4-x} + \ln 2$$

$$\left(= \ln 2|3x+2| - e^{4-x}\right)$$

Your notes

5.4.2 Techniques of Integration

Integrating Composite Functions (ax+b)

What is a composite function?

- A **composite function** involves one function being applied after another
- A composite function may be described as a "function of a function"
- This Revision Note focuses on one of the functions being **linear** – i.e. of the form $\boldsymbol{ax+b}$

How do I integrate linear (*ax+b)* functions?

- A **linear function** (of x) is of the form $ax+b$
- The special cases for **trigonometric functions** and **exponential** and **logarithm functions** are
  - $\int \sin(ax+b)\ \mathrm{d}x = -\frac{1}{a}\cos(ax+b)+c$
  - $\int \cos(ax+b)\ \mathrm{d}x = \frac{1}{a}\sin(ax+b)+c$
  - $\int \mathrm{e}^{ax+b}\ \mathrm{d}x = \frac{1}{a}\mathrm{e}^{ax+b}+c$
  - $\int \frac{1}{ax+b}\ \mathrm{d}x = \frac{1}{a}\ln|ax+b|+c$
- There is one more special case
  - $\int (ax+b)^n\ \mathrm{d}x = \frac{1}{a(n+1)}(ax+b)^{n+1}+c$ where $n \in \mathbb{Q},\ n \neq -1$
- c, in all cases, is the **constant** of **integration**
- All the above can be deduced using **reverse chain rule**
  - However, spotting them can make solutions more efficient

Examiner Tip

- Although the specific formulae in this revision note are NOT in the **formula booklet**
  - almost all of the information you will need to apply reverse chain rule is provided
  - make sure you have the formula booklet open at the right page(s) and practice using it

Your notes

Worked example

Find the following integrals

a) $\int 3(7-2x)^{\frac{5}{3}}\ \mathrm{d}x$

$I = \int 3(7-2x)^{5/3}\ dx = 3\int (-2x+7)^{5/3}\ dx$

Using $\int (ax+b)^n\ dx = \frac{1}{a(n+1)}(ax+b)^{n+1} + c$,

$I = 3\left[\frac{1}{-2\times\frac{8}{3}}(-2x+7)^{8/3}\right] + c$ ($\frac{8}{3}$ ← $\frac{5}{3}+1$)

$\therefore I = -\frac{9}{16}(7-2x)^{8/3} + c$

b) $\int \frac{1}{2}\cos(3x-2)\ \mathrm{d}x$

$I = \int \frac{1}{2}\cos(3x-2)\ dx = \frac{1}{2}\int \cos(3x-2)\ dx$

Using $\int \cos(ax+b)\ dx = \frac{1}{a}\sin(ax+b) + c$

$I = \frac{1}{2}\left[\frac{1}{3}\sin(3x-2)\right] + c$

$\therefore I = \frac{1}{6}\sin(3x-2) + c$

Your notes

Reverse Chain Rule

What is reverse chain rule?

- The **Chain Rule** is a way of differentiating two (or more) functions
- **Reverse Chain Rule** (RCR) refers to **integrating** by **inspection**
  - spotting that chain rule would be used in the reverse (differentiating) process

How do I know when to use reverse chain rule?

- **Reverse chain rule** is used when we have the **product** of a **composite function** and the **derivative** of its **secondary function**
- Integration is trickier than differentiation; many of the shortcuts do not work
  - For example, in general $\int e^{f(x)}\,\mathrm{d}x \neq \frac{1}{f'(x)}e^{f(x)}$
  - However, this result **is true** if $f(x)$ is linear $(ax+b)$
- Formally, in **function notation**, **reverse chain rule** is used for **integrands** of the form

$$I=\int g'(x)f'(g(x))\,\mathrm{d}x$$

  - this does not have to be strictly true, but '*algebraically*' it should be
    - if **coefficients** do not match '**adjust** and **compensate'** can be used
    - e.g. $5x^2$ is not quite the derivative of $4x^3$
      - the *algebraic* part (x^2) is '*correct*'
      - but the coefficient 5 is '*wrong*'
      - use '**adjust** and **compensate'** to '*correct*' it
- A particularly useful instance of reverse chain rule to recognise is

$$I=\int \frac{f'(x)}{f(x)}\,\mathrm{d}x=\ln|f(x)|+c$$

  - i.e. the **numerator** is (almost) the **derivative** of the **denominator**
  - '**adjust** and **compensate**' may need to be used to deal with any coefficients
    - e.g.

$$I=\int \frac{x^2+1}{x^3+3x}\,\mathrm{d}x=\frac{1}{3}\int 3\frac{x^2+1}{x^3+3x}\,\mathrm{d}x=\frac{1}{3}\int \frac{3x^2+3}{x^3+3x}\,\mathrm{d}x=\frac{1}{3}\ln|x^3+3x|+c$$

How do I integrate using reverse chain rule?

- If the product **can** be identified, the **integration** can be done "by **inspection**"
  - there may be some "**adjusting** and **compensating**" to do
- Notice a lot of the "**adjust** and **compensate** method" happens mentally
  - this is indicated in the steps below by quote marks

STEP 1
Spot the 'main' function

e.g. $I = \int x(5x^2 - 2)^6 \, dx$

"the main function is $(\ \dots\ )^6$ which would come from $(\ \dots\ )^7$"

STEP 2

'**Adjust**' and '**compensate**' any coefficients required in the integral

e.g. "$(\ \dots\ )^7$ would differentiate to $7(\ \dots\ )^6$"

"chain rule says multiply by the derivative of $5x^2 - 2$, which is $10x$"

"there is no '7' or '10' in the integrand so adjust and compensate"

$$I = \frac{1}{7} \times \frac{1}{10} \times \int 7 \times 10 \times x(5x^2 - 2)^6 \, dx$$

STEP 3

Integrate and simplify

e.g. $I = \frac{1}{7} \times \frac{1}{10} \times (5x^2 - 2)^7 + c$

$$I = \frac{1}{70}(5x^2 - 2)^7 + c$$

- Differentiation can be used as a means of checking the final answer
- After some practice, you may find Step 2 is not needed
  - Do use it on more awkward questions (negatives and fractions!)
- If the product **cannot** easily be identified, use **substitution**

Examiner Tip

- Before the exam, practice this until you are confident with the pattern and do not need to worry about the formula or steps anymore
  - This will save time in the exam
- You can always check your work by differentiating, if you have time

Your notes

Your notes

Worked example

A curve has the gradient function $f'(x) = 5x^2\sin(2x^3)$.

Find an expression for $f(x)$.

$$f(x) = \int 5x^2 \sin(2x^3)\,dx$$

$$f(x) = 5\int x^2 \sin(2x^3)\,dx$$ Take 5 out as a factor

This is a product, almost in the form $g'(x)f(g(x))$

STEP 1: Spot the 'main' function

"the main function is sin (...) which would come from cos (...)"

STEP 2: 'Adjust and compensate' coefficients

"cos (...) would differentiate to −sin (...)"
"$2x^3$ would differentiate to $6x^2$"

$$f(x) = 5 \times - \times \frac{1}{6} \times \int - \times 6 \times x^2 \sin(2x^3)\,dx$$

compensate (5 × − × 1/6), adjust (− × 6)

STEP 3: Integrate and simplify

$$f(x) = -\frac{5}{6}\cos(2x^3) + c$$

Your notes

Substitution: Reverse Chain Rule

What is integration by substitution?

- When reverse chain rule is difficult to spot or awkward to use then **integration** by **substitution** can be used
  - **substitution** simplifies the integral by defining an alternative variable (usually u) in terms of the original variable (usually x)
  - **everything** (including "$\mathrm{d}x$" and **limits** for **definite integrals**) is then substituted which makes the integration much easier

How do I integrate using substitution?

STEP 1
Identify the substitution to be used – it will be the secondary function in the composite function

So $g(x)$ in $f(g(x))$ and $u = g(x)$

STEP 2
Differentiate the substitution and rearrange

$\frac{\mathrm{d}u}{\mathrm{d}x}$ can be treated like a fraction

(i.e. "multiply by $\mathrm{d}x$" to get rid of fractions)

STEP 3
Replace all parts of the integral
All x terms should be replaced with equivalent u terms, including $\mathrm{d}x$
If finding a **definite integral** change the limits from x-values to u-values too

STEP 4
Integrate and either

substitute x back in

or

evaluate the defínte integral using the u limits (either using a GDC or manually)

STEP 5
Find c, the constant of integration, if needed

- For **definite integrals**, a GDC should be able to process the integral without the need for a substitution
  - be clear about whether working is required or not in a question

Examiner Tip

- Use your GDC to check the value of a definite integral, even in cases where working needs to be shown

Your notes

Your notes

Worked example

a) Find the integral

$$\int \frac{6x+5}{(3x^2+5x-1)^3}\,dx$$

STEP 1: Identify the substitution

The composite function is $(3x^2+5x-1)^3$

The secondary function of this is $3x^2+5x-1$

$\therefore$ Let $u = 3x^2+5x-1$

STEP 2: Differentiate u and rearrange

$$\frac{du}{dx} = 6x+5$$

$$\therefore\ du = (6x+5)\,dx$$

STEP 3: Replace all parts of the integral

$$I = \int \frac{6x+5}{(3x^2+5x-1)^2}\,dx = \int \frac{du}{u^3}$$

$$= \int u^{-3}\,du$$

STEP 4: Integrate and substitute x back in

(STEP 5 not needed, evaluating c is not required)

$$I = \frac{u^{-2}}{-2} + c$$

$$I = -\frac{1}{2}(3x^2+5x-1)^{-2} + c$$

$$\therefore\ I = \frac{-1}{2(3x^2+5x-1)^2} + c$$

b) Evaluate the integral

$$\int_1^2 \frac{6x+5}{(3x^2+5x-1)^3}\,\mathrm{d}x$$

giving your answer as an exact fraction in its simplest terms.

Note that you could use your GDC for this part
Certainly use it to check your answer!

From STEP 3 above, $I = \int_{x=1}^{x=2} u^{-3}\,du$

Change limits too, $x=1,\ u=3(1)^2+5(1)-1=7$
$x=2,\ u=3(2)^2+5(2)-1=21$

STEP 4: Integrate and evaluate

$$I=\left[-\tfrac{1}{2}u^{-2}\right]_7^{21} = \left[-\tfrac{1}{2}(21)^{-2}\right]-\left[-\tfrac{1}{2}(7)^{-2}\right]$$

$$\therefore I = \frac{4}{441}$$

Your notes

Your notes

5.4.3 Definite Integrals

Definite Integrals

What is a definite integral?

$$\int_a^b f(x)\ dx = [F(x)]_a^b = F(b) - F(a)$$

- This is known as the **Fundamental Theorem of Calculus**
- ***a*** and ***b*** are called limits
  - ***a*** is the lower limit
  - ***b*** is the upper limit
- $f(x)$ is the **integrand**
- $F(x)$ is an **antiderivative** of $f(x)$
- The **constant** of **integration** ("**+c**") is not needed in **definite integration**
  - "+c" would appear alongside both **F(*a*)** and **F(*b*)**
  - subtracting means the "+c"'s cancel

How do I find definite integrals analytically (manually)?

STEP 1
Give the integral a name to save having to rewrite the whole integral every time
If need be, rewrite the integral into an integrable form

$$I = \int_a^b f(x)\ dx$$

STEP 2
Integrate without applying the limits; you will not need "+c"
Notation: use square brackets [ ] with limits placed at the end bracket

STEP 3
Substitute the limits into the function and evaluate

Examiner Tip

- If a question does not state that you can use your GDC then you must show all of your working clearly, however it is always good practice to check you answer by using your GDC if you have it in the exam

Your notes

Worked example

a) Show that

$$\int_2^4 3x(x^2-2)\,dx = 144$$

STEP 1: Name the integral and rewrite into an integratable form

$$I = \int_2^4 (3x^3 - 6x)\,dx$$

STEP 2: Integrate

$$I = \left[\frac{3}{4}x^4 - 3x^2\right]_2^4$$

STEP 3: Evaluate

$$I = \left[\frac{3}{4}(4)^4 - 3(4)^2\right] - \left[\frac{3}{4}(2)^4 - 3(2)^2\right]$$

$$I = 144 - 0$$

$$\therefore \int_2^4 3x(x^2-2) = 144$$

b) Use your GDC to evaluate

$$\int_0^1 3e^{x^2 \sin x}\,dx$$

giving your answer to three significant figures.

Using GDC,

$$\int_0^1 3e^{x^2 \sin x}\, dx = 3.872\,957\ldots$$

$$\therefore \int_0^1 3e^{x^2 \sin x}\, dx = 3.87 \quad (3 \text{ s.f.})$$

Your notes

Your notes

Properties of Definite Integrals

Fundamental Theorem of Calculus

$$\int_a^b f(x)\ dx = [F(x)]_a^b = F(b) - F(a)$$

- Formally,
  - $f(x)$ is **continuous** in the interval $a \le x \le b$
  - $F(x)$ is an **antiderivative** of $f(x)$

What are the properties of definite integrals?

- Some of these have been encountered already and some may seem obvious ...
  - taking **constant** factors outside the integral
    - $\int_a^b kf(x)\ dx = k\int_a^b f(x)\ dx$ where k is a constant
    - useful when fractional and/or negative values involved
  - integrating term by term
    - $\int_a^b [f(x) + g(x)]\ dx = \int_a^b f(x)\ dx + \int_a^b g(x)\ dx$
    - the above works for subtraction of terms/functions too
  - equal upper and lower limits
    - $\int_a^a f(x)\ dx = 0$
    - on evaluating, this would be a value, subtract itself !
  - swapping limits gives the same, but negative, result
    - $\int_a^b f(x)\ dx = -\int_b^a f(x)\ dx$
    - compare 8 subtract 5 say, with 5 subtract 8 ...
  - splitting the interval
    - $\int_a^b f(x)\ dx = \int_a^c f(x)\ dx + \int_c^b f(x)\ dx$ where $a \le c \le b$
    - this is particularly useful for areas under multiple curves or areas under the x-axis
  - horizontal translations
    - $\int_a^b f(x)\ dx = \int_{a-k}^{b-k} f(x+k)\ dx$ where k is a constant
    - the graph of $y = f(x \pm k)$ is a horizontal translation of the graph of $y = f(x)$ ($f(x+k)$ translates left, $f(x-k)$ translates right)

Examiner Tip

- Learning the properties of definite integrals can help to save time in the exam

Your notes

Your notes

Worked example

$f(x)$ is a continuous function in the interval $5 \le x \le 15$.

It is known that $\int_5^{10} f(x)\,dx = 12$ and that $\int_{10}^{15} f(x)\,dx = 5$.

a) Write down the values of

i) $\int_7^7 f(x)\,dx$

ii) $\int_{10}^5 f(x)\,dx$

i. $\int_7^7 f(x)\,dx = 0$

"equal limits"

$\int_a^a f(x)\,dx = 0$

ii. $\int_{10}^5 f(x)\,dx = -12$

"swapped limits"

$\int_b^a f(x)\,dx = -\int_a^b f(x)\,dx$

b) Find the values of

i) $\int_5^{15} f(x)\,dx$

ii) $\int_5^{10} 6f(x+5)\,dx$

Your notes

i. $I = \int_5^{15} f(x)\,dx = \int_5^{10} f(x)\,dx + \int_{10}^{15} f(x)\,dx = 12 + 5 = 17$

"split limits"

$$\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^a f(x)\,dx$$

$$\therefore \int_5^{15} f(x)\,dx = 17$$

ii. $I = \int_5^{10} 6f(x+5) = 6\int_5^{10} f(x+5)\,dx$

"factors"

$$\int_a^b kf(x)\,dx = k\int_a^b f(x)\,dx$$

$$I = 6\int_{5+5}^{10+5} f(x)\,dx$$

"horizontal translation"

$$\int_a^b f(x+k)\,dx = \int_{a+k}^{b+k} f(x)\,dx$$

$$I = 6\int_{10}^{15} f(x)\,dx = 6 \times 5 = 30$$

$$\therefore \int_5^{10} 6f(x+5)\,dx = 30$$

Your notes

5.4.4 Further Applications of Integration

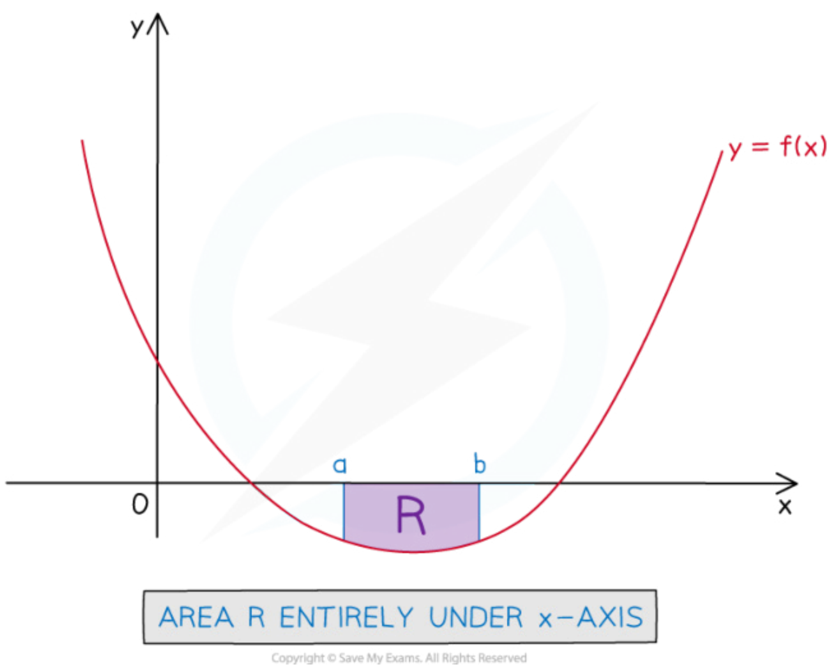
Negative Integrals

- The area under a curve may appear **fully** or **partially** under the x-axis
  - This occurs when the function $f(x)$ takes **negative** values within the boundaries of the area
- The **definite integrals** used to find such **areas**
  - will be **negative** if the area is **fully** under the x-axis
  - possibly **negative** if the area is **partially** under the x-axis
    - this occurs if the negative area(s) is/are greater than the positive area(s), their **sum** will be **negative**
- When using a GDC use the modulus (absolute value) function so that all definite integrals have a positive value

$$A = \int_a^b |y| \, dx$$

  - This is given in the **formula booklet**

How do I find the area under a curve when the curve is fully under the x-axis?



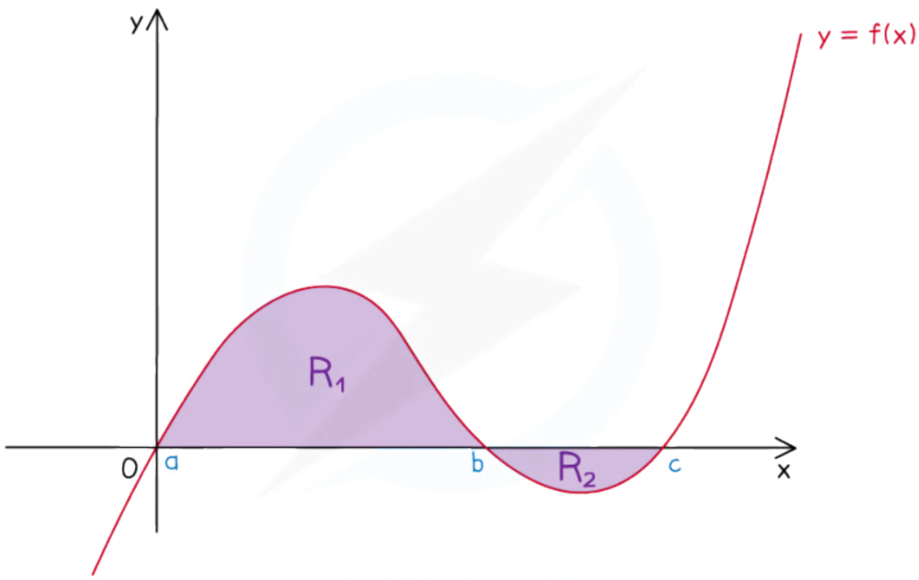
STEP 1
Write the expression for the definite integral to find the area as usual

This may involve finding the lower and upper limits from a graph sketch or GDC and f(x) may need to be rewritten in an integrable form

STEP 2
The answer to the definite integral will be negative
Area must always be positive so take the modulus (absolute value) of it
e.g. If $I = -36$ then the area would be 36 (square units)

Your notes

How do I find the area under a curve when the curve is partially under the x-axis?



- For questions that allow the use of a GDC you can still use

$$A = \int_a^c |f(x)| \, dx$$

- To find the area analytically (manually) use the following method
  STEP 1
  Split the area into parts - the area(s) that are above the x-axis and the area(s) that are below the x-axis

  STEP 2
  Write the expression for the definite integral for each part (give each part a name, I_1, I_2, etc)
  This may involve finding the lower and upper limits of each part from a graph sketch or a GDC, finding the roots of the function (i.e. where $f(x) = 0$) and rewriting $f(x)$ in an integrable form

  STEP 3

Your notes

Find the value of each definite integral separately

STEP 4
Find the area by summing the modulus (absolute values) of each integral
(Mathematically this would be written $A = |I_1| + |I_2| + |I_3| + \ldots$)

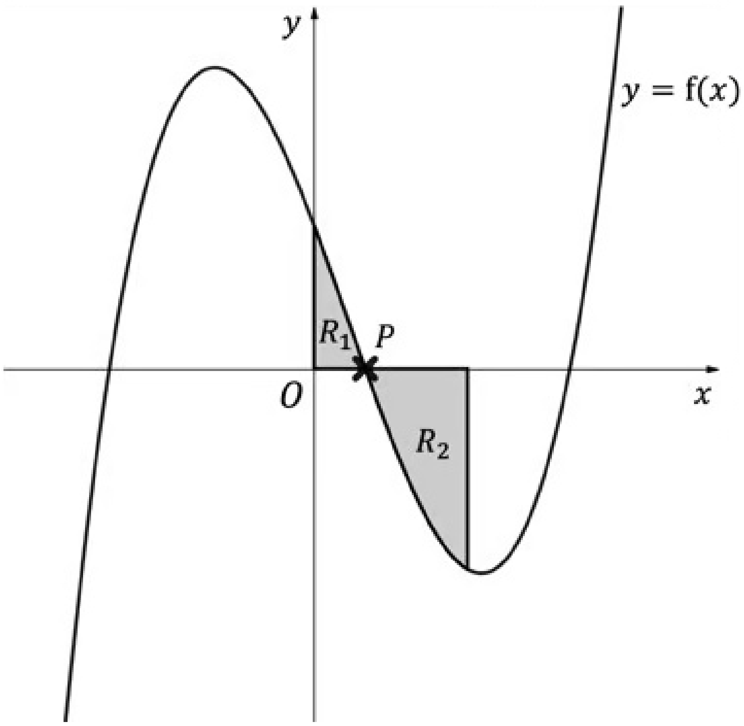
Examiner Tip

- If no diagram is provided, quickly sketch one so that you can see where the curve is above and below the x - axis and split up your integrals accordingly

Your notes

Worked example

The diagram below shows the graph of $y = f(x)$ where $f(x) = (x+4)(x-1)(x-5)$.



The region R_1 is bounded by the curve $y = f(x)$, the x-axis and the y-axis.

The region R_2 is bounded by the curve $y = f(x)$, the x-axis and the line $x = 3$.

a) Determine the coordinates of the point labelled P.

a) The x-coordinate of P is a root of $f(x)$

$f(x) = 0$

$(x+4)(x-1)(x-5) = 0$

$x = -4, \; x = 1, \; x = 5$

Clearly from the graph, $x = 1$ at point P

$\therefore P(1, 0)$

b) i) Find a definite integral that would help find the area of the shaded region R_2 and briefly explain why this would **not** give the area of the region R_2.

ii) Find the exact area of the shaded region R_2.

Your notes

b) i)

$$I_2 = \int_1^3 (x+4)(x-1)(x-5)\,dx$$

R_2 is underneath the x-axis so the value of the definite integral will be negative. Area cannot be negative.

ii) STEP 1:

$$I_2 = \int_1^3 (x+4)(x-1)(x-5)\,dx$$

$$I_2 = \int_1^3 (x^2+3x-4)(x-5)\,dx$$ Rewrite in an integrable form

$$I_2 = \int_1^3 (x^3-2x^2-19x+20)\,dx$$

$$I_2 = \left[\frac{x^4}{4} - \frac{2x^3}{3} - \frac{19x^2}{2} + 20x\right]_1^3$$ Integrate (no need for "+c")

$$I_2 = \left(\frac{3^4}{4} - \frac{2(3)^3}{3} - \frac{19(3)^2}{2} + 20(3)\right) - \left(\frac{1}{4} - \frac{2}{3} - \frac{19}{2} + 20\right)$$

$$I_2 = -\frac{93}{4} - \frac{121}{12}$$

$$I_2 = -\frac{100}{3}$$

STEP 2: $\therefore$ Area of R_2, $A_2 = \frac{100}{3}$ square units

c) Find the exact total area of the shaded regions, R_1 and R_2.

c) STEP 1, 2: $A_1 = I_1 = \int_0^1 (x^3-2x^2-19x+20)\,dx$ Use the relevant results from b) ii)

$$I_1 = \left[\frac{x^4}{4} - \frac{2x^3}{3} - \frac{19x^2}{2} + 20x\right]_0^1$$

STEP 3: $I_1 = \frac{121}{12} - 0$

STEP 4: $\therefore A_1 + A_2 = \frac{121}{12} + \frac{100}{3} = \frac{521}{12}$

$\therefore$ Total area shaded $= \frac{521}{12}$ square units

You can check the final answer using your GDC and the formula (in booklet) $A = \int_a^b |y|\,dx$.

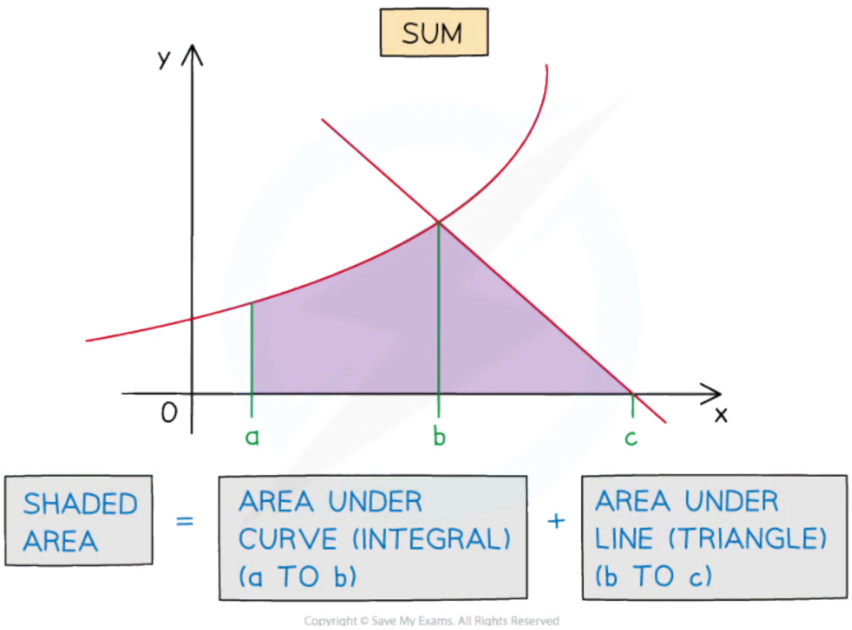
Here, $A = \int_0^3 |(x+4)(x-1)(x-5)|\,dx$

$A = 43.41666...$

(Note that our GDC was not able to produce the exact answer...)

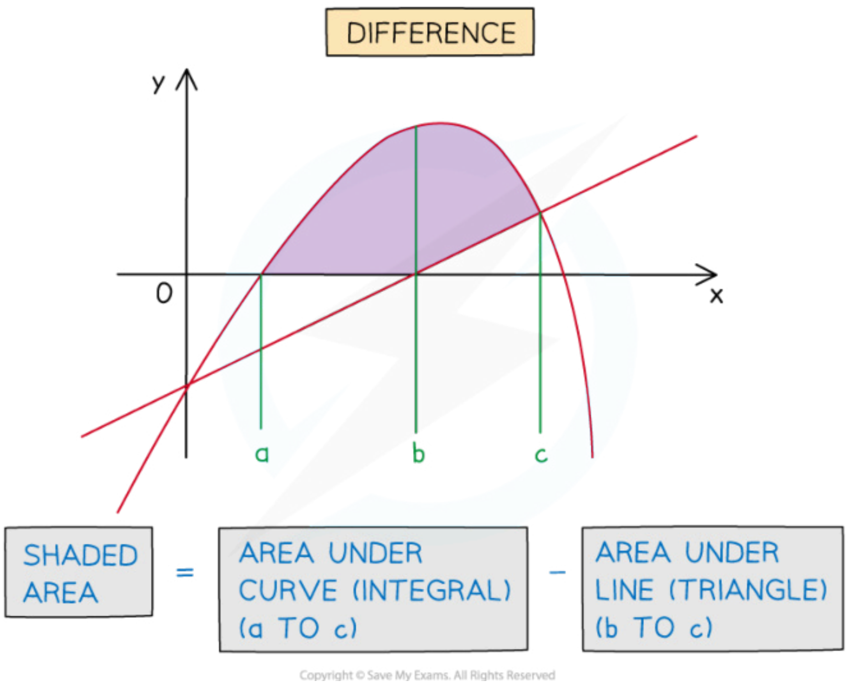
Area Between a Curve and a Line

- **Areas** whose boundaries include a **curve** and a (non-vertical) **straight line** can be found using integration
  - For an **area** under a **curve** a **definite integral** will be needed
  - For an **area** under a **line** the shape formed will be a **trapezium** or **triangle**
    - **basic area formulae** can be used rather than a definite integral
    - (although a definite integral would still work)
- The **area** required could be the **sum** or **difference** of areas under the curve and line



Your notes

Your notes



How do I find the area between a curve and a line?

STEP 1
If not given, sketch the graphs of the curve and line on the same diagram
Use a GDC to help with this step

STEP 2
Find the intersections of the curve and the line
If no diagram is given this will help identify the area(s) to be found

STEP 3
Determine whether the area required is the sum or difference of the area under the curve and the area under the line
Calculate the area under a curve using a integral of the form

$$\int_a^b y \, \mathrm{d}x$$

Calculate the area under a line using either $A = \frac{1}{2}bh$ for a triangle or $A = \frac{1}{2}h(a+b)$ for a trapezium (y-coordinates will be needed)

STEP 4
Evaluate the definite integrals and find their sum or difference as necessary to obtain the area required

Examiner Tip

- Add information to any diagram provided
- Add axes intercepts, as well as intercepts between lines and curves
- Mark and shade the area you're trying to find
- If no diagram is provided, **sketch** one!

Your notes

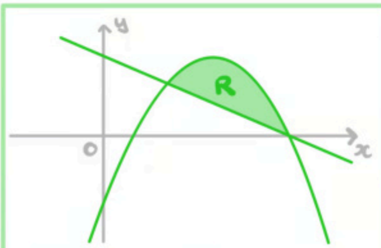
Worked example

The region R is bounded by the curve with equation $y = 10x - x^2 - 16$ and the line with equation $y = 8 - x$.

R lies entirely in the first quadrant.

a) Using your GDC, or otherwise, sketch the graphs of the curve and the line on the same diagram. Identify and label the region R on your sketch and use your GDC to find the x-coordinates of the points of intersection between the curve and the line.

STEP 1:



STEP 2: Points of intersection are $(3, 5)$ and $(8, 0)$

b) i) Write down an integral that would find the area of the region R.

ii) Find the area of the region R.

i) STEP 3: Curve is 'upper' boundary of R

$\therefore\ y_1 = 10x - x^2 - 16$

$y_2 = 8 - x$

$y_1 - y_2 = 10x - x^2 - 16 - (8 - x) = 11x - x^2 - 24$

$\therefore$ Area of R, $A_R = \int_3^8 (11x - x^2 - 24)\, dx$

ii) STEP 4: $A_R = \int_3^8 (11x - x^2 - 24)\, dx$

$A_R = \left[\frac{11x^2}{2} - \frac{x^3}{3} - 24x \right]_3^8$

$A_R = \left[\frac{11(8)^2}{2} - \frac{(8)^3}{3} - 24(8) \right] - \left[\frac{11(3)^2}{2} - \frac{(3)^3}{3} - 24(3) \right]$

$A_R = -\frac{32}{3} - -\frac{63}{2}$

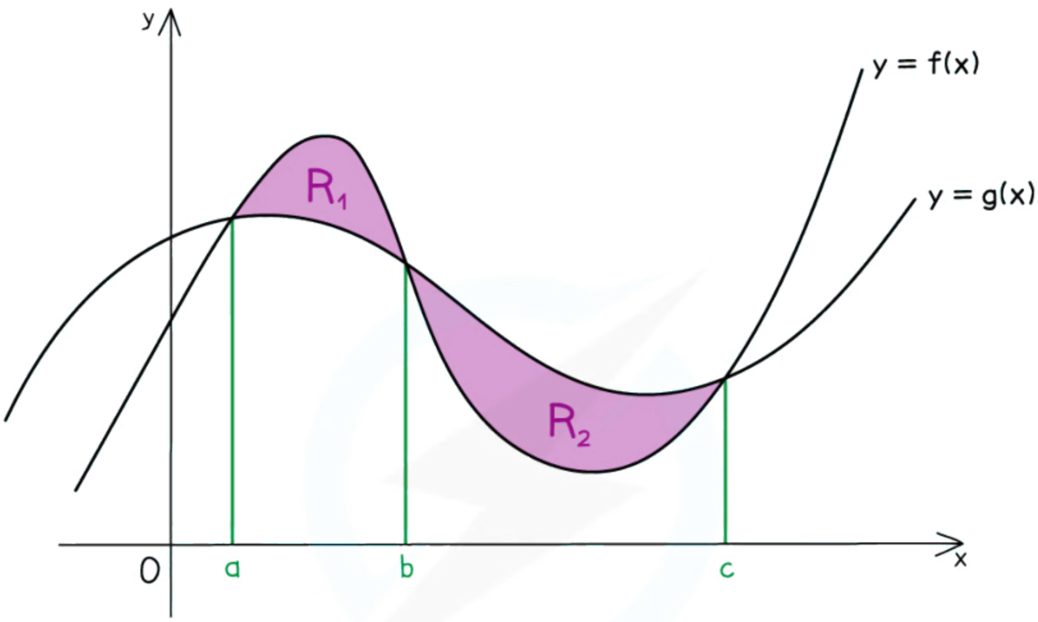
$\therefore$ Area of region R is $\frac{125}{6}$ square units

Your notes

Your notes

Area Between 2 Curves

- **Areas** whose boundaries include **two curves** can be found by integration
  - The **area between two curves** will be the **difference** of the areas under the two curves
    - both areas will require a **definite integral**
  - Finding points of intersection may involve a more awkward equation than solving for a curve and a line



$$R_1 = \int_a^b [f(x) - g(x)]dx$$

$$R_2 = \int_b^c [g(x) - f(x)]dx$$

How do I find the area between two curves?

STEP 1
If not given, sketch the graphs of both curves on the same diagram
Use a GDC to help with this step

STEP 2
Find the intersections of the two curves
If no diagram is given this will help identify the area(s) to be found

Your notes

STEP 3
For each area (there may only be one) determine which curve is the 'upper' boundary
For each area, write a definite integral of the form

$$\int_a^b (y_1 - y_2)\, \mathrm{d}x$$

where y_1 is the function for the 'upper' boundary and y_2 is the function for the 'lower' boundary
Be careful when there is more than one region – the 'upper' and 'lower' boundaries will swap

STEP 4
Evaluate the definite integrals and sum them up to find the total area
(Step 3 means no definite integral will have a negative value)

Examiner Tip

- If no diagram is provided sketch one, even if the curves are not accurate
- Add information to any given diagram as you work through a question
- Maximise use of your GDC to save time and maintain accuracy:
  - Use it to sketch the graphs and help you visualise the problem
  - Use it to find definite integrals

Your notes

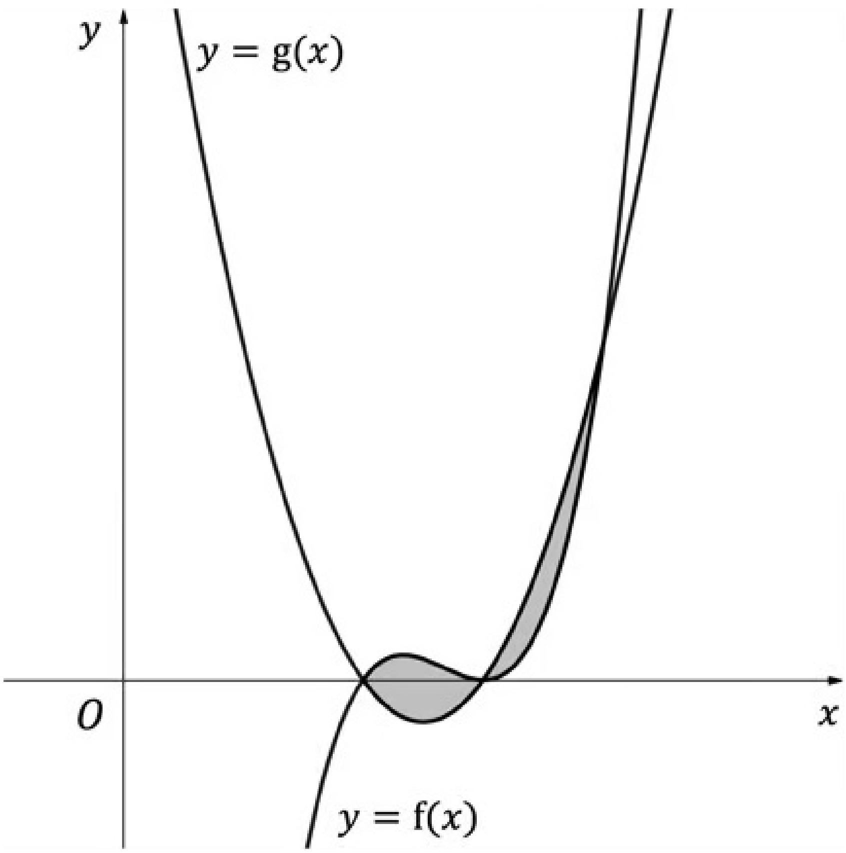
Worked example

The diagram below shows the curves with equations $y = \mathrm{f}(x)$ and $y = g(x)$ where

$$\mathrm{f}(x) = (x-2)(x-3)^2$$

$$\mathrm{g}(x) = x^2 - 5x + 6$$

Find the area of the shaded region.



Your notes

STEP 1: Sketch of graph given

STEP 2: Two intersections are the roots of $f(x)$

$f(x) = (x-2)(x-3)^2 = 0$ at $x = 2$, $(y=0)$

and $x = 3$ $(y=0)$

Solve $f(x) = g(x)$ to find the other intersection

$(x-2)(x-3)^2 = x^2 - 5x + 6$

$(x-2)(x-3)^2 = (x-2)(x-3)$

$x - 3 = 1$

$x = 4$, $y = (4-2)(4-3) = 2$

STEP 3: The area, A_1 of the first region is given by

$A_1 = \int_2^3 \left[(x-2)(x-3)^2 - (x^2-5x+6)\right] dx$

$A_1 = \int_2^3 (x-2)(x-3)\left[(x-3)-1\right] dx$ Factorise $(x-2)(x-3)$

$A_1 = \int_2^3 (x^2-5x+6)(x-4)\ dx$

$A_1 = \int_2^3 (x^3 - 9x^2 + 26x - 24)\ dx$

$A_1 = \left[\frac{x^4}{4} - 3x^3 + 13x^2 - 24x\right]_2^3$

$A_1 = \left(\frac{(3)^4}{4} - 3(3)^3 + 13(3)^2 - 24(3)\right) - \left(\frac{(2)^4}{4} - 3(2)^3 + 13(2)^2 - 24(2)\right)$

$A_1 = -\frac{63}{4} - -16 = \frac{1}{4}$

For A_2, the 'upper' and 'lower' boundaries swap

$A_2 = \int_3^4 \left[(x^2-5x+6) - (x-2)(x-3)^2\right] dx$

$A_2 = \int_3^4 (x-2)(x-3)\left[1-(x-3)\right] dx$

$A_2 = \int_3^4 (x^2-5x+6)(4-x)\ dx$

$A_2 = \int_3^4 (-x^3 + 9x^2 - 26x + 24)\ dx$

$A_2 = \left[-\frac{x^4}{4} + 3x^3 - 13x^2 + 24x\right]_3^4$

$A_2 = \left(-\frac{(4)^4}{4} + 3(4)^3 - 13(4)^2 + 24(4)\right) - \left(-\frac{(3)^4}{4} + 3(3)^3 - 13(3)^2 + 24(3)\right)$

$A_2 = 16 - \frac{63}{4} = \frac{1}{4}$

Total area is $A_1 + A_2$

∴ Area of shaded region is $\frac{1}{2}$ square unit